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DYNAMIC NON-LINEAR ELECTROMAGNETIC WAVE PROPAGATION AND HARMONIC RADIATION IN MAGNETO-IONIC MEDIA

by
O. E. H. Rydbeck

June 1, 1963

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Ionosphere Research Laboratory

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I. Introduction and Summary

In a recent Scientific Report, "Electromagnetic Non-Linear Wave Interaction and Reflection from a Plane Ionised Medium" [3], the author has studied non-linear wave interaction and its associated wave generations (at various combination frequencies) in an isotropic, ionized medium. In order to make the results as general as possible, the interaction and the non-linear reflection laws were expressed in terms of unspecified refractive indices. This made it possible, among other things, to express the conditions for travelling wave resonance in vector form, which allows a study of the interaction, for example the generation of sum or difference frequency radiation, between two waves which have arbitrary, and different, angles of incidence upon the medium.

In the present report the non-linear propagation studies are extended to a magneto-ionic medium; the limitation being, that all wave normals have been assumed to be parallel. The general theory, involving arbitrary wave normals, will be dealt with in a forthcoming report.

It is shown that, as expected, travelling wave resonances occur in the magneto-ionic medium, not only between waves of the same kind (e.g. between waves of ordinary polarization) but equally frequently between waves of opposite kinds. From the non-linear interaction point of view there, thus, is not much difference, at least in principle, between the ordinary and extraordinary waves. One also finds, which is of interest in this connection, that the second order non-linear waves are characterized by two different kinds of polarization which become equal only at travelling wave resonance.

Travelling wave resonances should be very efficient in the "top-side" ionosphere, where both losses and electron density gradients are small. It is interesting to note, that most of them take place in regions where $X > 1-Y$ (in the usual notation); which are accessible by both types of waves from a "top-side" sounder. Interaction instabilities are likely to be strong in regions near or close to fourth reflection level conditions (where the electron velocities become very large), especially for $Y^2=1$, and $Y^2=1-X$. Parametric harmonic pumping of cyclotron-type resonances are particularly probable - in agreement with recent experimental results.

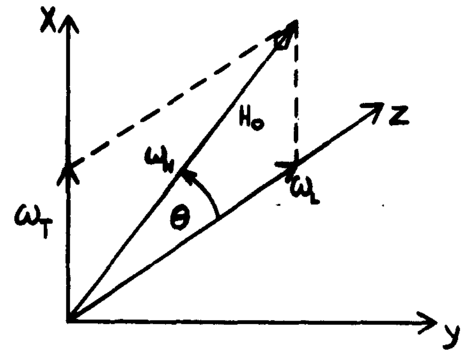
Under certain conditions second harmonic (echo) generation also is possible at or near the following combined fundamental and second harmonic reflection levels, viz. $X=1,2$ and 4. Similar levels can be established for the higher harmonics.

The author expresses the hope that future "top-side" sounders will also be equipped with harmonic sweep recording devices. No doubt this would greatly extend our knowledge of dynamic, non-linear "top-side" phenomena.

In conclusion, it should be added that the results, obtained in this report, can also be used to study and evaluate parametric amplification, or second harmonic generation, of exospheric whistler modes.

II. The Electron Velocities and the Differential (AC) Space Charge Density of the Linear Magneto-Ionic Wave

Let us assume that the static magnetic field, of strength H_o, and cyclotron frequency, ω_H, is oriented with respect to the wave normal (z-direction) and the coordinate system as shown in Fig. 1.



(direction of wave normal, angular wave frequency ω_q)

Fig. 1 Demonstrating orientation of the static magnetic field

If we denote the linear fields, of angular frequency ω_q, as follows, $E_y = E_{y_q}^{(1)}$, etc., the transverse, linear electron velocities can be written [1].

$$v_{q_y}^{(1)} = -\frac{1-n_q^2}{X_q}\frac{eE_{q_y}^{(1)}}{j\omega_q m}\ ; \quad v_{q_x}^{(1)} = -\frac{1-n_q^2}{X_q}\frac{eE_{q_x}^{(1)}}{j\omega_q m}\ , \left(\frac{\partial}{\partial x} = 0 = \frac{\partial}{\partial y}\right) \tag{II,1}$$

where n_q is the refractive index of the wave in question, and

$X_q = \omega_p^2/\omega_q^2$, where ω_p is the angular (electronic) plasma frequency of the ionized medium. Introducing the cyclotron frequency ratio, $Y_q = \omega_H/\omega_q$, its transverse and longitudinal components, $Y_{T_q} = Y_q \sin\theta_q$, $Y_{L_q} = Y_q \cos\theta$, and the plasma resonance factor

$$R_q = \frac{1}{1 - X_q - j\delta_q} , \quad (\delta_q = \nu/\omega_q) \qquad \text{(II,2)}$$

the Appleton-Hartree expression for n_q^2 assumes the convenient form

$$1 - n_{q_2}^2 = \frac{X_q}{1 - j\delta_q - \frac{1}{2} Y_{T_q}^2 R_q \pm \sqrt{\frac{1}{4} Y_{T_q}^4 R_q^2 + Y_{L_q}^2}} , \qquad \text{(II,3)}$$

which demonstrates the influence of the transversely coupled plasma oscillation upon the propagation of the linear wave.

The longitudinal velocity component can be written

$$v_{q_z}^{(1)} = Y_{T_q} R_q \frac{1 - n_q^2}{X_q} \frac{e E_{q_y}^{(1)}}{\omega_q m} , \qquad \text{(II,4)}$$

and is, thus, in phase with $E_{q_y}^{(1)}$ for a loss-less medium. Rel. (II,4) by (II,1) also can be written

$$v_{q_z}^{(1)} = -jY_{T_q} R_q v_{q_y}^{(1)} . \qquad \text{(II,4a)}$$

$v^{(1)}_{q_z}$ and $v^{(1)}_{q_y}$ are consequently of the same order of magnitude ($Y_{q_T} \neq 0$) <u>except</u> near the plasma resonance level, where R_q may be very large. In this region, therefore, $v^{(1)}_{q_z}$ may give rise to appreciable non-linear effects (provided that $\delta_q^2 \ll 1$) even if the transverse velocities remain small.

The ac space charge density can be written [1]

$$\frac{\rho_{1_q}}{\rho_o} = \frac{\Delta N_q}{N_o} = \frac{v^{(1)}_{q_z}}{c_o/n_q} = \frac{v^{(1)}_{q_z}}{v_{phase_q}}$$

, ($\rho_o = -eN_o$; N_o is the mean electron density, and c_o the electromagnetic velocity in vacuum)

(II,5)

which by (II,4a) can be given the alternate, and in this case important form,

$$\frac{\rho_{1_q}}{\rho_o} = -jY_{T_q} \left\{ R_q \frac{v^{(1)}_{q_y}}{v_{ph_q}} \right\} . \qquad \text{(II,5a)}$$

ρ_{1_q} thus is zero when $Y_T = 0$ (longitudinal propagation), and when $R_q n_q = 0$. Relation (II,5a) furthermore shows that, as far as the charge bunching is concerned, the measure of non-linearity is $R_q v^{(1)}_{q_y}/v_{ph_q}$, and <u>not</u> $v^{(1)}_{q_y}/v_{ph_q}$. From (II,5) it also appears that ρ_{1_q}/ρ_o may become very large for the extra-ordinary wave (the Z-component) at or very near the plasma resonance level.

According to (II,1), (II,4) and (II,5) we can regard the magneto-ionic wave as composed of a transverse electromagnetic wave, represented by $v_y^{(1)}$ and $v_x^{(1)}$, and an associated plasma wave, represented by $v_z^{(1)}$ and $\rho_1^{(1)}$, which are coupled to each other by the transverse magnetic field. This plasma wave disappears when $\theta=0$, and therefore does not appear in the linear theory, when the propagation is purely longitudinal. The plasma wave may become very large at the plasma resonance level and is physically instrumental in producing the triple split coupling.

In the linear magneto-ionic theory one only considers the linear convection current density $\rho_o \bar{v}_q^{(1)}$. At the plasma resonance level, however, one may also have to consider the non-linear term $\rho_q^{(1)} \bar{v}_q^{(1)}$; which produces a second harmonic. The moment the ionised medium drifts one must also deal with the linear drift component $\rho_q^{(1)} \bar{v}_o$, where $\bar{v}_o$ is the drift velocity. Different types of waves, such as space charge waves, may then be generated in the medium. A discussion thereof is outside the scope of the present communication. For details the reader is referred to a paper by J. Askne and the author, on electron stream whistler mode interaction [2].

To make the linear field survey complete, we should perhaps recall that

$$H_{q_y}^{(1)} = \frac{n_q}{Z_o} E_{q_x}^{(1)} \; ; \quad H_{q_x}^{(1)} = - \frac{n_q}{Z_o} E_{q_y}^{(1)} \, , \qquad \text{(II,6)}$$

and

$$E_{q_x}^{(1)}/E_{q_y}^{(1)} = S_{q_{\stackrel{o}{x}}}^{(1)} = jQ_{q_{\stackrel{o}{x}}}^{(1)} = -jY_{L_q} \frac{1}{-\frac{1}{2}Y_{T_q}{}^2 R_q \pm \sqrt{\frac{1}{4}Y_{T_q}{}^4 R_q{}^2 + Y_{L_q}{}^2}} =$$

$$= jY_{L_q} R_q \frac{1-n_q{}^2}{1-R_q n_q{}^2(1-j\delta_q)} , \qquad \text{(II,7)}$$

which is the linear (or first order) polarization ratio. The polarization of the non-linear field components is generally different from $S_q^{(1)}$; see [1], and (IV,13).

In the study of non-linear effects one is naturally interested in regions where the linear velocity fields, and for that matter also $\rho_1^{(1)}$, generated by a high power wave, for example a pump wave, become very large. We have seen that $v_z^{(1)}$ and $\rho_1^{(1)}$ behave in this manner at or near the plasma resonance level, whereas the other components remain small.

It appears from our previous relation that there is another, and in this connection very important resonance level, when <u>all</u> velocities and the ac space charge density may become very large (in a medium with small losses) and that is at or near the so called fourth reflection level, where $|n_q{}^2| \rightarrow \infty$. It is well known, as is easily verified by (II,3), that this "resonance" takes place when

$$Y_q^{\ 2} = Y_{R_q}^{\ 2} = \frac{1-X_q}{1-X_q\cos^2\theta_q} , \qquad (II,8)$$

or expressed in X_q, when

$$X_q = X_{R_q} = \frac{1-Y_q^{\ 2}}{1-Y_q^{\ 2}\cos^2\theta_q} . \qquad (II,8a)$$

When X_q is small, (II,8) approximately yields

$$Y_{R_q}^{\ 2} \simeq 1 - X_q\sin^2\theta_q \quad ,$$

i.e. one is very close to "cyclotron resonance"; especially when θ_q is small. The latter case is especially important when one considers the excitation of cyclotron harmonics (or near harmonics) in the "topside" ionosphere.

It is interesting to note that (II,8) can also be written

$$\underbrace{(\omega_q^{\ 2}-\omega_H^{\ 2})}_{\substack{\text{cyclotron}\\ \text{osc. term}}} \ \underbrace{(\omega_q^{\ 2}-\omega_p^{\ 2})}_{\substack{\text{plasma}\\ \text{osc. term}}} = \underbrace{\omega_H^{\ 2}\omega_p^{\ 2}\sin^2\theta}_{\substack{\text{coupling}\\ \text{term}}} = \omega_T^{\ 2}\omega_p^{\ 2} , \qquad (II,9)$$

which shows that the fourth reflection resonance can be considered as a transverse coupling between the plasma and cyclotron oscillations.

Let us, in conclusion, summarize the results of this Section. As far as the linear driving forces are concerned, we have two important resonance levels in the ionised medium, viz.

$$X_q = 1 \text{ , and } X_q = \frac{1-Y_q^{\,2}}{1-Y_q^{\,2}\cos^2\theta} .$$

Besides these resonance there are, of course, non-linear travelling wave resonances [3]. These will be discussed in Section IV.

III. The Non-Linear Driving Forces

If we retain the non-linear terms in the equation of motion, we have

$$\frac{\partial \bar{v}}{\partial t} = -\frac{e}{m}\bar{E} + \frac{e}{m}\mu_0\left[(\bar{H}_0+\bar{H}) \times \bar{v}\right] - (\bar{v}\cdot\nabla)\bar{v} = -\frac{e}{m}\bar{E} + \bar{\omega}_H x \bar{v} + \bar{F} , \qquad \text{(III,1)}$$

where $\bar{F}$ is the non-linear force on the oscillating electrons. These are assumed to have zero drift motion in the linear theory (a stationary magneto-ionic medium).

If we, for the moment, consider self-nonlinearities only (for example with a powerful primary wave) the second order force term can be written in the instructive form

$$\bar{F}^{(2)} = -\frac{1}{2}\,\text{grad}\,(|\bar{v}^{(1)}|^2) + \left[\bar{v}^{(1)} \times \text{curl}\;\; \bar{v}^{(1)} + \int^t \frac{e}{m}\bar{E}^{(1)}dt)\right]. \qquad \text{(III,2)}$$

Since $-\mu_0 \frac{\partial \bar{H}}{\partial t} = \text{curl}\ \bar{E}$, this relation can also be written

$$\bar{F}^{(2)} = -\frac{1}{2}\,\text{grad}\,(|\bar{v}^{(1)}|^2) + \left[\bar{v}^{(1)} x\ (\text{curl}\ \bar{v}^{(1)} - \bar{\omega}_{H_1}^{(1)})\right] , \qquad \text{(III,2a)}$$

where $\bar{\omega}_{H_1}^{(1)}$ is the "ac" cyclotron frequency. This yields the following basic relation for the isotropic non-linear medium, viz.

$$\bar{p}^{(2)} = -\frac{1}{2}\,\mathrm{grad}\,(|\bar{v}^{(1)}|^2)\ . \quad (H_o=0) \qquad \text{(III,2b)}$$

In this case the (second order) non-linear force is proportional to the gradient of the <u>first order kinetic energy</u> of the oscillating electrons.

* * *

If we next assume that there are two waves present in the medium, i.e. q = 1,2, it can be shown [1] that $\bar{p}^{(2)}$ has the following components (ν=0):

$$p_z^{(2)} = \underbrace{-\frac{1}{2}\frac{\partial}{\partial z}\left[v_{z_1}^{(1)^2} + \frac{X_1}{1-n_1^2}\left(v_{y_1}^{(1)^2} + v_{x_1}^{(1)^2}\right)\right]}_{p_{z_1}^{(2)}} \underbrace{-\frac{1}{2}\frac{\partial}{\partial z}\left[v_{z_2}^{(1)^2} + \frac{X_2}{1-n_2^2}\left(v_{y_2}^{(1)^2} + v_{x_2}^{(1)^2}\right)\right]}_{p_{z_2}^{(2)}} -$$

$$\underbrace{-\frac{\partial}{\partial z}\left(v_{z_1}^{(1)} v_{z_2}^{(1)}\right) - \frac{X_1}{1-n_1^2}\left[\frac{\partial v_{y_1}^{(1)}}{\partial z} v_{y_2}^{(1)} + \frac{\partial v_{x_1}^{(1)}}{\partial z} v_{x_2}^{(1)}\right] - \frac{X_2}{1-n_2^2}\left[\frac{\partial v_{y_2}^{(1)}}{\partial z} v_{y_1}^{(1)} + \frac{\partial v_{x_2}^{(1)}}{\partial z} v_{x_1}^{(1)}\right]}_{p_{z_{12}}^{(2)}}, \qquad \text{(III,3)}$$

$$p_y^{(2)} = \underbrace{-\frac{1}{2} v_{z_1}^{(1)} \frac{\partial v_{y_1}^{(1)}}{\partial z}\left(1-\frac{X_1}{1-n_1{}^2}\right)}_{p_{y_1}^{(2)}} \underbrace{-\frac{1}{2} v_{z_2}^{(1)} \frac{\partial v_{y_2}^{(1)}}{\partial z}\left(1-\frac{X_2}{1-n_2{}^2}\right)}_{p_{y_2}^{(2)}} -$$

$$\underbrace{-\, v_{z_1}^{(1)} \frac{\partial v_{y_2}^{(1)}}{\partial z}\left(1-\frac{X_2}{1-n_2{}^2}\right) - v_{z_2}^{(1)} \frac{\partial v_{y_1}^{(1)}}{\partial z}\left(1-\frac{X_1}{1-n_1{}^2}\right)}_{p_{y_{12}}^{(2)}} , \qquad (III,4)$$

$$p_x^{(2)} = \underbrace{-\frac{1}{2} v_{z_1}^{(1)} \frac{\partial v_{x_1}^{(1)}}{\partial z}\left(1-\frac{X_1}{1-n_1{}^2}\right)}_{p_{x_1}^{(2)}} \underbrace{-\frac{1}{2} v_{z_2}^{(1)} \frac{\partial v_{x_2}^{(1)}}{\partial z}\left(1-\frac{X_2}{1-n_2{}^2}\right)}_{p_{x_2}^{(2)}} -$$

$$\underbrace{-\, v_{z_1}^{(1)} \frac{\partial v_{x_2}^{(1)}}{\partial z}\left(1-\frac{X_2}{1-n_2{}^2}\right) - v_{z_2}^{(1)} \frac{\partial v_{x_1}^{(1)}}{\partial z}\left(1-\frac{X_1}{1-n_1{}^2}\right)}_{p_{x_{12}}^{(2)}} . \qquad (III,5)$$

It should be noted that complex values of the electron velocities cannot be used in these relations. The real velocity values are easily obtained from the various field relations in the previous Section.

In the case of the ionosphere it normally is sufficient to use the second order relations (III,3,4,5). In a physical plasma device this is not always permissible. In this case, however, the second order relations still are very useful, since they can be used to demonstrate how and when wave instabilities tend to build-up [3].

It is important to note that the transverse non-linear forces ($P_y^{(2)}$, and $P_x^{(2)}$) disappear, if $v_{z_{1,2}}^{(1)} = 0$. If all waves travel longitudinally, there are second order non-linear driving forces in the longitudinal (z) direction only. This case, therefore, is simpler to treat, but much less interesting from the interaction point of view.

The $\bar{F}_1^{(2)}$, and $\bar{F}_2^{(2)}$ terms contain the second harmonic forces, of angular frequencies $2\omega_1$, and $2\omega_2$, and a non-linear, static force term. Since $v_z^{(1)}$ differs in phase from $v_y^{(1)}$ by $\pm \pi/2(\nu=0)$, it appears from (III,3,4,5) that there is a second order static, non-linear force term in the y-direction only. As the total "dc"-current must be zero in the infinite plane medium, we must require that the mean value, denoted by $\langle\ \rangle$, of the second order convection current be zero, i.e.

$$\langle \rho_{1_{\frac{1}{2}}}^{(1)} \; \bar{v}_{\frac{1}{2}}^{(1)} + \rho_0 \; \bar{v}_{\frac{1}{2}}^{(2)} \rangle = 0 \; , \qquad \text{(III,6)}$$

which, by the $\bar{v}$-and ρ_1-relations of Section II, yields

$$\left.\begin{aligned}&\langle v^{(2)}_{x_{\substack{1\\2}}}\rangle = -R_{\substack{1\\2}} Y_{T_{\substack{1\\2}}} Q^{(1)}_{\substack{1\\2}} \frac{1}{2}\left(\frac{v^{(1)}_{y_{max}}}{v_{ph}}\right)^2_{\substack{1\\2}} ; \langle v^{(2)}_{z_{\substack{1\\2}}}\rangle = \langle v^{(2)}_{x_{\substack{1\\2}}}\rangle R_{\substack{1\\2}} Y_{T_{\substack{1\\2}}} / Q^{(1)}_{\substack{1\\2}}\\ &\langle v^{(2)}_{y_{\substack{1\\2}}}\rangle = 0 \quad .\end{aligned}\right\}(\nu = 0) \qquad (III,7)$$

Since $\langle v_{z_{\substack{1\\2}}}\rangle$ always is positive, the higher power wave, to second order, pushes the medium ahead. There is no transverse drift and, it should be noted, that $\langle \bar{v}^{(1)}\rangle$ has the same direction (in the x-z plane) as the linear electron velocity.

From the second order equation of motion we next obtain

$$\left.\begin{aligned}&\frac{e}{m}\langle E^{(2)}_x\rangle = -\omega_L\langle v^{(2)}_y\rangle + \langle P^{(2)}_x\rangle \; ; \; \frac{e}{m}\langle E^{(2)}_y\rangle = \omega_L\langle v^{(2)}_x\rangle - \omega_T\langle v^{(2)}_z\rangle + \langle P^{(2)}_y\rangle\\ &\frac{e}{m}\langle E^{(2)}_z\rangle = \omega_T\langle v^{(2)}_y\rangle + \langle P^{(2)}_z\rangle \quad . \text{ (Indices 1 and 2 dropped)}\end{aligned}\right\} \qquad (III,8)$$

This yields

$$\langle E^{(2)}_{x_{\substack{1\\2}}}\rangle = 0 = \langle E^{(2)}_{z_{\substack{1\\2}}}\rangle \; ; \; \frac{e}{\omega_{\substack{1\\2}} m}\langle E^{(2)}_{y_{\substack{1\\2}}}\rangle = -R_{\substack{1\\2}} Y_{T_{\substack{1\\2}}}\left[Y_{L_{\substack{1\\2}}}\left(Q^{(1)}_{\substack{1\\2}} - \frac{1}{Q^{(1)}_{\substack{1\\2}}}\right) + R_{\substack{1\\2}} Y^2_{T_{\substack{1\\2}}}\right] \frac{1}{2}\left(\frac{v^{(1)}_{y_{max}}}{v_{ph}}\right)^2_{\substack{1\\2}} \quad . \qquad (III,9)$$

Thus a static, non-linear electric field is generated in the y-direction only. It has the right magnitude and direction to permit the electrons to drift in a straight line in spite of the presence of the static magnetic field.

Finally, it should be noted that, in a system with one high power wave only,

$$p_x^{(2)}/p_y^{(2)} = \frac{\partial v_x^{(1)}}{\partial z}\Bigg/\frac{\partial v_y^{(1)}}{\partial z} \quad . \qquad \text{(III,10)}$$

* * *

Besides the non-linear force on the electrons we must also consider the non-linear "ac" convection current densities in the system, viz.

$$\bar{i}_a^{(2)} = \rho_{1_1}^{(1)}\bar{v}_1^{(1)} + \rho_{1_2}^{(1)}\bar{v}_2^{(1)} + \rho_{1_1}^{(1)}\bar{v}_2^{(1)} + \rho_{1_2}^{(1)}\bar{v}_1^{(1)} - \langle i_a^{(2)}\rangle , \qquad \text{(III,9)}$$

and

$$\bar{I}_b^{(2)} = \rho_o(\bar{v}_1^{(2)} + \bar{v}_2^{(2)}) + \langle i_a^{(2)}\rangle \quad ,$$

which also contain second harmonics and sum and difference frequency terms.

IV. The Non-Linear Wave Equation

For the second order (non-linear) fields Maxwell's equations can now be written

$$\left.\begin{aligned} &p_e{}^2 E_x^{(2)} = -\frac{1}{\varepsilon_o}\frac{\partial}{\partial t}\left(\bar{i}_a^{(2)} + \bar{I}_b^{(2)}\right)_x ; \quad p_e{}^2 E_y = -\frac{1}{\varepsilon_o}\frac{\partial}{\partial t}\left(\bar{i}_a^{(2)} + \bar{I}_b^{(2)}\right)_y ; \\ &\frac{\partial^2}{\partial t^2} E_z^{(2)} = -\frac{1}{\varepsilon_o}\frac{\partial}{\partial t}\left(\bar{I}_a^{(2)} + \bar{I}_b^{(2)}\right)_z \quad , \end{aligned}\right\} \text{(IV,1)}$$

where

$$p_e^2 = -\frac{1}{\mu_o \varepsilon_o}\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2} \quad \text{(Electromagnetic wave operator in vacuum)} \quad (IV,2)$$

If we introduce the additi onal operators, viz.

$$p_p^2 = \frac{\partial^2}{\partial t^2} + \omega_p^2 \quad \text{(Plasma oscillation, operator)} \qquad p_{e_m}^2 = p_e^2 + \omega_p^2 \quad ,(IV,3)$$

(Electromagnetic wave operator for the isotropic ionized medium)

and make us of the various expressions for the $\mathbf{P}^{(2)}$-components of the previous Section, we obtain (after some transformations) the following non-linear wave equation for $v_y^{(2)}$, viz.

$$Dv_y^{(2)} = p_p^2\left(\omega_L p_e^2 \psi_x + p_{em}^2 \frac{\partial \psi_y}{\partial t}\right) - \omega_T p_{em}^2 p_e^2 \psi_z = \Pi_d \quad . \quad (IV,4)$$

where

$$\left.\begin{aligned} \psi_x &= p_e^2 P_x^{(2)} - \omega_p^2 \frac{\partial W_x^{(2)}}{\partial t} \; ; \quad \psi_y = p_e^2 P_y^{(2)} - \omega_p^2 \frac{\partial W_y^{(2)}}{\partial t} \\ \psi_z &= \frac{\partial^2}{\partial t^2} P_z^{(2)} - \omega_p^2 \frac{\partial W_z^{(2)}}{\partial t} \quad . \end{aligned}\right\} (IV,5)$$

Here $W_x^{(2)}$, $W_y^{(2)}$, $W_z^{(2)}$ are the x, y, and z components of

$$\overline{W}^{(2)} = \frac{1}{\rho_o}\,\overline{i}_a^{(2)} \quad , \quad (IV,6)$$

and

$$D = p_{em}^2\left(p_p^2 p_{em}^2 + \omega_T^2 p_e^2\right)\frac{\partial^2}{\partial t^2} + \omega_L^2 p_p^2 p_e^4 \quad . \quad (IV,7)$$

One, furthermore, obtains

$$p_{em}^{\,2}\frac{\partial v_x^{(2)}}{\partial t} = -\,\omega_L p_e^{\,2} v_y^{(2)} + \psi_y \ , \qquad (IV,8)$$

and

$$p_p^{\,2}\frac{\partial v_z^{(2)}}{\partial t} = \omega_T \frac{\partial^2 v_y^{(2)}}{\partial t^2} + \psi_z \ . \qquad (IV,9)$$

Since $v_y^{(2)}$ is also related to $v_x^{(2)}$ by the following relation

$$\left(p_p^{\,2}p_{em}^{\,2}+\omega_T^{\,2}p_e^{\,2}\right)\frac{\partial^2 v_y^{(2)}}{\partial t^2} = \omega_L p_p^{\,2}p_e^{\,2}\frac{\partial v_x^{(2)}}{\partial t} - \omega_T p_e^{\,2}\psi_z + p_p^{\,2}\frac{\partial \psi_y}{\partial t} \ , \qquad (IV,10)$$

one is induced to introduce the two "symbolic" second order polarization ratios (compare (II,7))

$$s_1^{(2)} = \frac{\left(p_p^{\,2}p_{em}^{\,2}+\omega_T^{\,2}p_e^{\,2}\right)\partial/\partial t}{\omega_L p_p^{\,2}p_e^{\,2}} \ , \qquad (IV,11)$$

and

$$s_1^{(2)} = -\,\frac{\omega_L p_e^{\,2}}{p_{em}^{\,2}\,\partial/\partial t} \ . \qquad (IV,12)$$

By (IV,7) it can immediately be shown that

$$s_1^{(2)} = s_2^{(2)}\left(1 - \frac{D}{\omega_L^{\,2}p_p^{\,2}p_e^{\,4}}\right) \ . \qquad (IV,13)$$

The two second order polarization ratios are therefore <u>only</u> equal when $D = 0$, i.e. when we have resonances in the system.

By our previous relations the wave equation for $v_x^{(2)}$

now can be written

$$D\,\frac{\partial v_x^{(2)}}{\partial t} = \left(p_p^{\,2}p_{em}^{\,2}+\omega_T^{\,2}p_e^{\,2}\right)\frac{\partial^2\psi_x}{\partial t^2} - \omega_L p_e^{\,2}p_p^{\,2}\,\frac{\partial\psi_y}{\partial t} + \omega_L\omega_T p_e^{\,4}\psi_z\,, \tag{IV,14}$$

which by (IV,11) and (IV,12) transforms to

$$Dv_x^{(2)} = p_p^{\,2}\left(\omega_L p_e^{\,2}S_1^{(2)}\psi_x + p_{em}^{\,2}S_2^{(2)}\,\frac{\partial\psi_y}{\partial t}\right) - \omega_T p_{em}^{\,2}p_e^{\,2}S_2^{(2)}\psi_z\,. \tag{IV,15}$$

This relation demonstrates the meaning of the two polarization terms: $S_1^{(2)}$ applies to the (non-linear) ψ_x forces, and $S_2^{(2)}$ to the ψ_y and ψ_z forces. Therefore $v_x^{(2)}/v_y^{(2)} = S_1^{(2)} = S_2^{(2)}$, when we have resonance in the system.

The wave equation for the second order, longitudinal velocity finally becomes

$$D\,\frac{\partial v_z^{(2)}}{\partial t} = \omega_T\left(\omega_L p_e^{\,2}\,\frac{\partial^2\psi_x}{\partial t^2} + p_{em}^{\,2}\,\frac{\partial^3\psi_y}{\partial t^3}\right) + \overbrace{\left(p_{em}^{\,4}\,\frac{\partial^2}{\partial t^2} + \omega_L^{\,2}p_e^{\,4}\right)}^{\alpha}\psi_z\,, \tag{IV,16}$$

where it is of interest to note that α annihilates the longitudinal magneto-ionic modes.

One finds from (IV,7) that D can be written

$$p_p^{\,2}D = \left\{p_e^{\,2}\left(p_p^{\,2}+\frac{\omega_T^{\,2}}{2}\right) + p_p^{\,2}\omega_p^{\,2}\right\}^2\frac{\partial^2}{\partial t^2} - p_e^{\,4}\left(\frac{\omega_T^{\,4}}{4}\,\frac{\partial^2}{\partial t^2} - \omega_L^{\,2}p_p^{\,4}\right), \tag{IV,17}$$

whence one symbolically obtains

$$p_p^{\,2}D = \underbrace{p_p^{\,2}\left[\left\{p_e^{\,2}\left(1+\frac{\omega_T^{\,2}/2}{p_p^{\,2}}\right)+\omega_p^{\,2}\right\}\frac{\partial}{\partial t} - p_e^{\,2}\sqrt{\left(\frac{\omega_T^{\,2}/2}{p_p^{\,2}}\right)^2\frac{\partial^2}{\partial t^2}-\omega_L^{\,2}}\,\right]}_{D_o} \times$$

Ordinary Electromagnetic Wave and Plasma Oscillation Operator

$$\times \underbrace{p_p^{\,2}\left[\left\{p_e^{\,2}\left(1+\frac{\omega_T^{\,2}/2}{p_p^{\,2}}\right)+\omega_p^{\,2}\right\}\frac{\partial}{\partial t} + p_e^{\,2}\sqrt{\left(\frac{\omega_T^{\,2}/2}{p_p^{\,2}}\right)^2\frac{\partial^2}{\partial t^2}-\omega_L^{\,2}}\,\right]}_{D_x} .$$

Extraordinary Electromagnetic Wave Operator

(IV,18)

$$p_p^{\,2}D = D_oD_x \quad . \tag{IV,18a}$$

It should be noted that D_o annihilates both an ordinary wave and a plasma oscillation ($p_p^{\,2} = 0$), whereas D_x annihilates only an extraordinary wave.

Subject to proper boundary conditions, $D_oD_x\bar{v}^{(2)} = 0$ yields the linear transients of the system and its self oscillations ($\nu = 0$, $\partial/\partial z = 0$) $X = \omega_p^{\,2}/\omega^2 = 1 + Y$, 1, and $1 - Y$.

* * *

If we assume that we have two high power waves, represented by $\bar{v}_1^{(1)}$ and $\bar{v}_2^{(1)}$, present in the system, the non-linear driving forces ψ_x, ψ_y, and ψ_z may contain terms with the following propagation factors, viz.

$$\left.\begin{array}{l}\exp.(2\omega_1 t\pm 2\omega_1 n_1 z/c_o);\ \exp.(2\omega_2 t\pm 2\omega_2 n_2 z/c_o);\ (n_1=n(\omega_1);\ n_2=n(\omega_2))\\ \exp.\left\{(\omega_1+\omega_2)t\pm(\omega_1 n_1+\omega_2 n_2)z/c_o\right\};\ \exp.\left\{(\omega_1+\omega_2)t\pm(\omega_1 n_1-\omega_2 n_2)z/c_o\right\};\\ \exp.\left\{(\omega_1-\omega_2)t\pm(\omega_1 n_1-\omega_2 n_2)z/c_o\right\};\ \exp.\left\{(\omega_1-\omega_2)t\pm(\omega_1 n_1+\omega_2 n_2)z/c_o\right\}.\end{array}\right\} \quad (IV,19)$$

It should be remembered in this connection that a primary high power wave is defined as a "pure" magneto-ionic mode. Two different magneto-ionic modes have to be treated as two primary waves, even if their wave frequency happens to be the same.

According to (IV,9) the "effective" refractive indices of the second harmonic driving forces is $2n(\omega_1)$ and $2n(\omega_2)$. The effective refractive indices, n_+ and n_-, of the sum and difference frequency waves, become

$$\left.\begin{array}{l} n_+ = \dfrac{\omega_1 n_1 \pm \omega_2 n_2}{\omega_1+\omega_2} = \dfrac{\omega_1 n_1 \pm \omega_2 n_2}{\omega_+}\ ,\\[2ex] n_- = \dfrac{\omega_1 n_1 \mp \omega_2 n_2}{\omega_1-\omega_2} = \dfrac{\omega_1 n_1 \mp \omega_2 n_2}{\omega_-}\ ,\end{array}\right\} \quad (IV,20)$$

where the upper signs correspond to the situation that both primary waves propagate in the same direction, and the lower signs to propagation in opposite directions.

Next, let us introduce the (self explanatory) operator equivalents

$$p_{e_\pm}^{\ 2} = \omega_\pm^{\ 2}(n_\pm^{\ 2}-1);\quad p_{em_\pm}^{\ 2} = p_{e_\pm}^{\ 2}+\omega_p^{\ 2}\ , \qquad (IV,21)$$

and

$$p_{p_\pm}^{\ 2} = \omega_p^{\ 2} - \omega_\pm^{\ 2} \quad . \qquad \text{(IV,22)}$$

Making use of these relations and (IV,20) one can prove that the following important relation holds, viz.

$$D_\pm = -p_{e_\pm}^{\ 2}\, p_{p_\pm}^{\ 2}\, \Gamma_\pm \quad , \qquad \text{(IV,23)}$$

where

$$\boxed{\Gamma_\pm = \omega_p^{\ 2} \left\{ \frac{1}{1-n_o^{\ 2}(\omega_\pm)} - \frac{1}{1-n_\pm^{\ 2}} \right\} \left\{ \frac{1}{1-n_x^{\ 2}(\omega_\pm)} - \frac{1}{1-n_\pm^{\ 2}} \right\}} \qquad \text{(IV,24)}$$

This relation shows that one obtains travelling wave resonance, $D = 0$, (compare reference [3]) in the system, when $n_\pm^{\ 2} = n_o^{\ 2}(\omega_\pm)$, or $n_\pm^{\ 2} = n_x^{\ 2}(\omega_\pm)$. In principle, therefore, the non-linear driving forces, irrespective of the type of polarization of the high power primary waves, may excite sum and/or difference frequency waves of either polarization. The only factor of importance (apart from the difference in magnitude of the non-linear driving forces in the various cases) is the travelling wave resonance, i.e. equality in phase velocity and in phase velocity direction, between the excited and the exciting waves. This is a characteristic feature of the (dynamic) non-linear magneto-ionic theory.

If collisional losses had been introduced in the system, we would have found that $\Gamma_\pm$ contained the same resonance terms but now with complex refractive indices. It is interesting to recall that maximum triple split coupling occurs when

$\{n_o(\omega) = n_x(\omega)\}_{X=1}$; which only is possible when collisional losses are introduced. Triple split coupling, therefore, can be regarded as a kind of travelling wave resonance in the coupling region. When $n_o(\omega_\pm) \approx n_x(\omega_\pm)$, $\Gamma_\pm$, and $D_\pm$, must be used with great care, since the non-linear driving forces now could excite both an o-wave and a x-wave at the same time. A detailed discussion of this complicated resonance case lies outside the scope of the present communication.

Let us focus our attention on (IV,4) for a moment. It appears from the previous relations, that $\psi_x = 0 = \psi_y$, if $\omega_T = 0$. In the case of purely longitudinal propagation no second order electromagnetic waves are excited in the system. Since, by (IV,7), $D = p_p^{\ 2}\alpha$ in this case, one finds from (IV,6) that a second order longitudinal (plasma) wave, determined by the relation

$$p_p^{\ 2} \frac{\partial v_z^{(2)}}{\partial t} = (\psi_z)_{\omega_T = 0} \ , \qquad (\omega_T = 0) \qquad\qquad (IV,25)$$

is excited in the system. Since $1 - n_q^{\ 2} = X_q$ for the isotropic medium ($\omega_H = 0$), (IV,25) yields a very simple result in this case, and $v_z^{(2)}$ (which becomes large when $p_p^{\ 2} \rightarrow 0$) is proportional to the gradient of the first order kinetic energy of the oscillating electrons [1] (see also (III,2b) and (III,3)). It consequently appears that no interesting second order non-linear interaction effects take place; unless $\omega_T \neq 0$.

The present theory has been based on the limiting assumption that the wave normals of the high power waves and the system waves are parallel. The more general case, when the wave normal directions differ by certain specified angles, will be discussed in a forthcoming report. One finds (see also [3]) that (IV,20) holds, provided it is written in vector form. If the medium is stratified in the z direction $\bar{n}_1$, $\bar{n}_2$, $\bar{n}_+$, and $\bar{n}_-$ in (IV,20) "simply" have to be replaced by their z-components.

V. Generation of Sum and Difference Frequency Waves in the Magneto-Ionic Medium

It appears from (IV,10) and (IV,24) that intense sum and difference frequency waves will be excited in the medium if: a) the high power (or pump) waves experience a medium resonance, i.e. if $X_{\substack{1\\2}} = 1$, or $Y_{\substack{1\\2}}^2 = Y_{R_{\substack{1\\2}}}^2$ ((II,8)), or b) if there is travelling wave resonance in the medium, i.e. $n_\pm^2 = n_o^2(\omega_\pm)$, or $n_\pm^2 = n_x^2(\omega_\pm)$. If both a) and b) happen at the same time (such cases are possible) one can expect extra strong excitation of the resonant waves.

For the homogeneous medium the associated sum and difference frequency waves (for nomenclature see also [3]) are easily obtained from (IV,10), when ψ_x, ψ_y and ψ_z have been evaluated by means of the various relations of Sections II and III. When the medium is inhomogeneous, which is the physically important case, the problem is more difficult. What one wishes to determine is not only the amplitude and phase of the associated waves but also the same quantities for the radiating sum and difference frequency waves.

However, it is possible to obtain an approximate, but useful, solution if the medium parameters vary very slowly, in fact so slowly that partial reflection can be neglected. To that end we proceed as follows.

It can easily be verified that $D_{\pm}$ can be written,

$$D_{\pm} = - \underbrace{\frac{\left(\omega_{\pm}^{2}-\omega_{p}^{2}\right)\left(\omega_{\pm}^{2}-\omega_{H}^{2}\right)-\omega_{T}^{2}\omega_{p}^{2}}{\omega_{\pm}^{2}-\omega_{p}^{2}}}_{\beta} c_{o}^{4} \left(\frac{d^{2}}{dz^{2}}+k_{o_{\pm}}^{2}\right)\left(\frac{d^{2}}{dz^{2}}+k_{x_{\pm}}^{2}\right), \quad (V,1)$$

where $k_{o_{\pm}} = \frac{\omega_{\pm}}{c_{o}} n_{o}(\omega_{\pm})$, and $k_{x_{\pm}} = \frac{\omega_{\pm}}{c_{o}} n_{x}(\omega_{\pm})$. It should be mentioned that the factor β, preceding the operators, is zero at the fourth reflection point (II,9) for an "independent" sum or difference frequency wave.

Let us assume, in order to study the radiation of the difference frequency wave only (the procedure to obtain the sum frequency wave is quite analogous), that Π_{d} of (IV,4) can be written

$$\Pi_{d} = b_{o}\, e^{+j\left\{\omega_{-}t-\frac{\omega_{-}}{c_{o}}\int_{z_{a}}^{z} n(\omega_{-})dz\right\}} = b_{o}\, e^{+j\left\{\omega_{-}t-\int_{z_{a}}^{z} k_{-}dz\right\}}, \quad (V,2)$$

where k_{-1} is considered to be positive, i.e. the non-linear driving force travels in the positive z-direction. The driving force amplitude, b_{o}, is assumed to vary so slowly (note:

resonances of the driving force, $X_{\frac{1}{2}} = 1$, or $Y^2_{\frac{1}{2}} = Y^2_{R_{\frac{1}{2}}}$, are assumed not to appear within the travelling wave resonance region - a matter that will be considered in a forthcoming report) that it can be regarded as constant in the main interaction region. Furthermore we assume that resonance takes place between the driving (non-linear) wave and the extraordinary wave of the system, and that $k_{o_-} \neq k_{x_-}$, i.e. we avoid the triple split region.

At some medium level, $z = z_o$, travelling wave resonance takes place. For the very slowly varying medium we, therefore, write

$$k_{-1} - k_{x_{-1}} = \varkappa(z - z_o) \quad , \tag{V,3}$$

which we assume to hold within the main interaction region, $z_o - z_m/2$, to $z_o + z_m/2$, where z_m is the width of the same.

If we next assert that z_m is so large, that

$$|w_m| = \left| \int_{z_o - z_m/2}^{z_o + z_m/2} (k_{-1} - k_{x_{-1}}) dz \right| \gg 1 \quad , \tag{V,4}$$

and make use of the asymptotic properties of the Fresnel integral [3], the first order solutions to (IV,4) become:

$$v_y^{(2)} \simeq \frac{b_o\, e^{+j\omega_- t}}{\beta(k_{o_-}{}^2 - k_-{}^2)(k_{x_-}{}^2 - k_-{}^2)}\, e^{-j\int_{z_a}^{z} k_{-1} dz} \,, \quad (z \approx z_o - z_m/2)$$

(V,5)

$$v_y^{(2)} \approx \frac{b_o\, e^{+j\omega_- t}}{\beta(k_{o_-}{}^2-k_-{}^2)(k_{x_-}{}^2-k_-{}^2)} \left\{ e^{-j\int_{z_a}^{z} k_{-1}\,dz} + \sqrt{\frac{2\pi}{\varkappa}}(k_{x_-}-k_-)\, e^{-j\left\{\int_{z_a}^{z} k_{x_-}\,dz - \frac{\pi}{4}\right\}} \right\} \qquad (V,6)$$

$$(z \approx z_o + z_m/2) \; .$$

At the bottom of the interaction region we have only one "associated" wave [3], which is not an independent (or radiating) wave, since it does not satisfy Maxwell's equations for the unperturbed magneto-ionic medium. At the top of the interaction region, $z \approx z_o + z_m/2$, we still obtain the same associated wave but also an "independent" radiating wave, $\exp.\left\{j(\omega_- t - \int_{z_a}^{z} k_{x_{-1}}\,dz\right\}$, which is a solution of Maxwell's equations, at the difference frequency, for the unperturbed magneto-ionic medium. We can regard the amplitude ratio, $\left|(k_- - k_{x_-})\sqrt{2\pi\varkappa}\right|$, between these two waves as a measure of the radiation efficiency (at the difference frequency).

Since the amplitude of the radiating wave, by (V,6), varies like $\sqrt{2\pi/\varkappa}$, we note that only a small gradient of the refractive index difference is required in order to generate a strong difference frequency radiation. Since $|w_m|$, of (V,4), must also be large, a very wide medium is required at the same time. If $|\varkappa| \rightarrow 0$, z_m must $\rightarrow \infty$ so fast, that (V,4) holds. Instabilities, therefore, to first order, develop only in a very extended interaction region.

Relation (V,6) can be used to evaluate the first order second harmonic radiation of a high power primary wave. According to the previous relations this takes place at the levels, where

$$n^2(\omega) = n_o^{\,2}(2\omega), \text{ or } n^2(\omega) = n_x^{\,2}(2\omega) \quad . \qquad (V,7)$$

Here $n(\omega)$ is the refractive index of the high power wave and, thus, can have index x or o, as the situation may be.

*　　　　*　　　　*

Next, let us investigate the possibilities of second harmonic travelling wave resonances according to (V,7). One immediately finds that, in the longitudinal case (Y_T=0),

$$Y_T^{\,2} = 0 \begin{cases} n_x^{\,2}(\omega)=n_x^{\,2}(2\omega)=1+\frac{X}{2}\ ,\ (X=\omega_p^{\,2}/\omega^2),\ \text{if } Y_L=\frac{\omega_L}{\omega}=3, & (V,8) \\ \\ n_o^{\,2}(\omega)=n_x^{\,2}(2\omega)=1-\frac{X}{2}\ ,\ \text{if } Y_L=\frac{\omega_L}{\omega}=1 & (V,9) \end{cases}$$

These resonances are particularly interesting since they hold for any X value and, therefore, are likely to generate strong second harmonics, in spite of the fact that the non-linear driving forces must be of the third order; when $Y_T^{\,2} = 0$. Case (V,9) corresponds to "cyclotron resonance" and should be observable; for example by top-side sounders.

Putting $Y_L = 0$, i.e. completely transverse propagation, one finds that

$$Y_T^2 = 0 \begin{cases} n_x^2(\omega) = n_x^2(2\omega) = 1 - \frac{X^2}{4}, \text{ if } Y_T^2 = \frac{\omega_T^2}{\omega^2} = -\frac{(1-X)(1-X/4)}{X/4} = Y_{T_{III}}^2, & (V,10) \\ \qquad\qquad \text{(Physically only possible if } 1 \leq X \leq 4) & \\ n_o^2(\omega) = n_x^2(2\omega) = 1 - X, \text{ if } Y_T^2 = 3(1 - X/4) = Y_{T_{IV}}^2 \ (X \leq 4) & (V,11) \end{cases}$$

and

$$n_x^2(\omega) = n_o^2(2\omega) = 1 - X/4, \text{ if } Y_T^2 = 3(X-1) = Y_{T_V}^2 \quad (X \geq 1) \qquad (V,12)$$

One further notices that (V,10) yields physically propagating, resonant waves only when $1 \leq X < 2$ $(0 \leq Y_{T_{III}}^2 < 1)$; (V,11) when $X < 4$ $(3 \geq Y_{T_{IV}}^2 > 0)$; and (V,11) when $1 \leq X < 4 (0 \leq Y_{T_V}^2 < 9)$. Thus, only case (V,11) is physically possible when $X < 1$.

The corresponding (travelling wave) resonance frequencies are:

$$\omega^2 = \frac{1}{8}\left(5\omega_p^2 \pm \sqrt{9\omega_p^4 - 16\omega_p^2\omega_T^2}\right) \quad (\omega_T^2 \leq \tfrac{9}{16}\,\omega_p^2) \qquad (V,10a)$$

$$\omega^2 = \frac{\omega_p^2}{4} + \frac{\omega_T^2}{3}, \qquad (V,11a)$$

$$\omega^2 = \omega_p^2 - \frac{\omega_T^2}{3}, \qquad (\omega_T^2 < 3\omega_p^2)$$

which do not always correspond to propagating conditions $(n^2(\omega) = n^2(2\omega) > 0)$.

It is interesting to note that it is also possible to obtain travelling wave resonances at, or near, the fourth reflection level where all electron velocities become very large. For this resonance to hold, we must require that $Y_R^2(\omega) = Y_R^2(2\omega)$, i.e. that

$$1 - X = \frac{Y_T^2}{1-Y_L^2}, \text{ and } 1 - \frac{X}{4} = \frac{Y_T^2/4}{1-Y_L^2/4},$$

which yields

$$\left.\begin{aligned} & Y_{L_R}^2 = \frac{4}{X}, \text{ and } Y_{T_R}^2 = \frac{(4-X)(X-1)}{X}, \quad (1 < X \leq 4) \\ & \text{i.e.} \\ & \qquad Y_R^2 = 5-X. \quad (Y_{T_{R_{max}}}^2 = 1, \text{ for } X = 2) \end{aligned}\right\} (|n^2(\omega)| = |n^2(2\omega)| = \infty) \qquad (V,13)$$

It is of interest to note, that this resonance only takes place when X lies in the range 1 to 4, and that Y_R^2 is never less than one. At these resonance levels an exospheric whistler may produce second harmonics. Conversely, a 2ω whistler "pump" wave may amplify (or generate) an ω whistler by parametric travelling wave interaction.

In the general case, where the wave normal makes an arbitrary angle with the static magnetic field, the refractive index relations become more complicated. Introducing the following parameter

$$\lambda = - Y_T^2/Y_{T_{III}}^2, \qquad (V,14)$$

the equalized refractive indices become

$$n_1^2(\omega) = n_1^2(2\omega) = 1 + \frac{X^2/2}{\lambda + \sqrt{\lambda^2+(\lambda+1)X^2}}, \qquad (V,15)$$

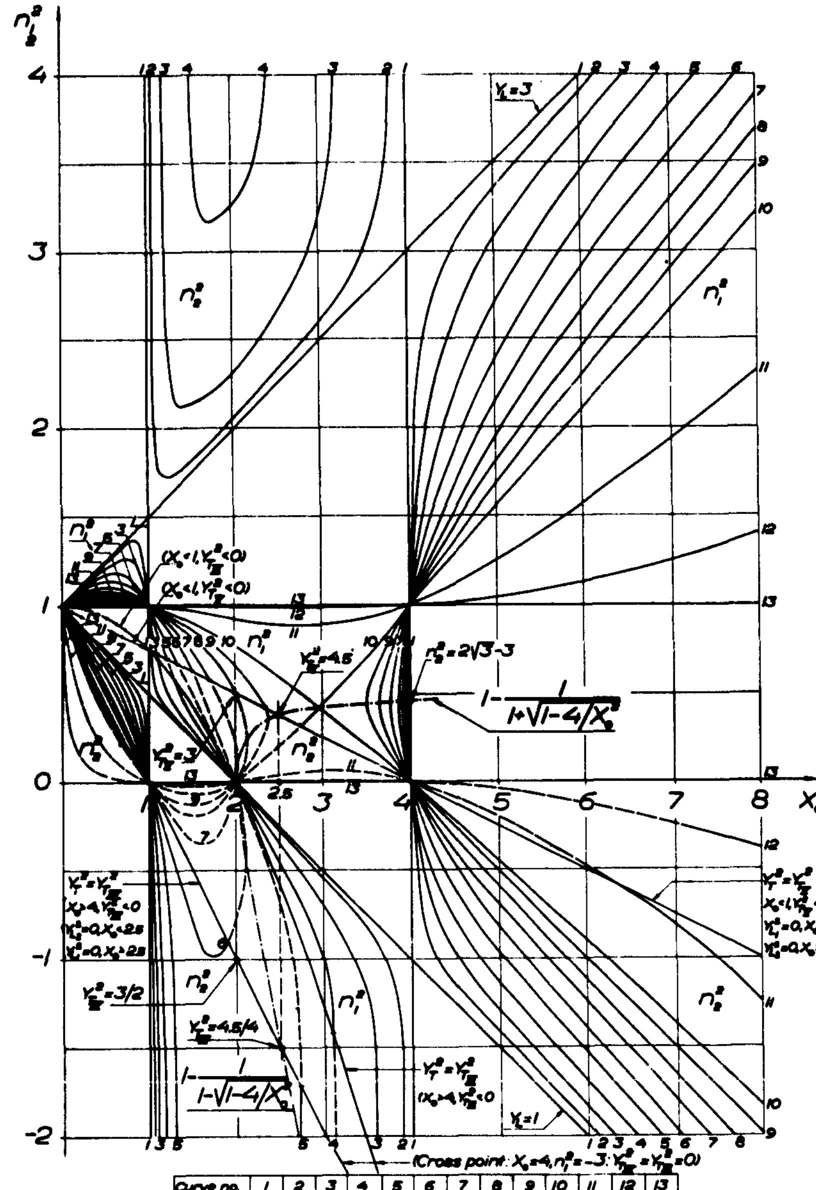
$$n_2^{\,2}(\omega) = n_2^{\,2}(2\omega) = 1 - \frac{X^2/2}{\underbrace{-\lambda+\sqrt{\lambda^2+(\lambda+1)X^2}}_{\mu}}\,, \qquad (V,16)$$

with the corresponding longitudinal cyclotron frequency components

$$Y_{L_{\frac{1}{2}}}^{\,2} = \frac{1}{X}\left\{2\lambda^2+5(\lambda+1)X\pm2(2+\lambda)\mu\right\} \;. \qquad (V,17)$$

For the purpose of graphical presentation we have labelled the equalized refractive indices 1 and 2. Only a more detailed investigation will reveal to which polarization the equalized indices may correspond; this varies with X and is of no immediate concern, since there is no principal difference between the various states of polarization from the point of view of non-linear interaction.

A closer examination of (V,15), and (V,16) reveals, that most of the travelling wave resonances take place in a region where $X > 1 - Y$. It is interesting to note from Fig. 2, which depicts the equalized refractive indices, (V,15) and (V,16) as functions of X, that there are three levels, viz. $X = 1,2$, and 4, for which $n^2(\omega) = 0 = n^2(2\omega)$. Generation of a fairly strong second harmonic echo can be anticipated at these levels. A detailed analysis of the radiation efficiency in this case, which must be based on a more rigorous wave treatment to replace (V,6), is outside the scope of the present communication. It should be noted, however, that the following



Curve no.	1	2	3	4	5	6	7	8	9	10	11	12	13
Y_T	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.30	4.00	10.00	∞

FIG. 2 DEMONSTRATING $n^2(\omega) = n^2(2\omega)$ AS FUNCTIONS OF $X = \omega_p^2/\omega^2$ (DENOTED X_0 IN FIG. 2), FOR VARIOUS VALUES OF Y_L AND Y_T

refractive indices are equal to zero at these levels, viz.

$$\left.\begin{aligned} &X = 1. \quad n_o(\omega) = n_x(2\omega), \text{ for } Y = 3/2 \ , \\ &X = 2. \quad n_z(\omega) = n_x(2\omega), \text{ for } Y = 1 \ , \\ &X = 4. \quad n_z(\omega) = n_o(2\omega), \text{ for } Y = 3 \ , \\ &\text{where } n_z(\omega) \text{ denotes } n_x(\omega), \text{ for } X > X_R \ . \end{aligned}\right\} \qquad (V,18)$$

According to Fig. 2 there are a number of levels, of special interest in the "top-side" ionosphere, where travelling wave excitation of second harmonics is possible. The same applies to the exosphere, where wave interaction takes place in regions where $Y > 1$. The different resonance levels may differ in efficiency, however, since the non-linear driving forces (represented by b_o in (V,5) may vary widely with X and Y.

VI. A Brief Discussion of the Non-Linear Interaction when Only the Effects of Pump Wave Charge Bunching is Considered

In order to discuss the parametric interaction in the magneto-ionic medium, let us limit ourselves to the effects of pump wave charge bunching only (see also [3]). We then have, ω_p is the angular pump frequency,

$$\psi_x = -\omega_p^2 \frac{\partial}{\partial t}(\eta v_x^{(2)}); \; \psi_y = -\omega_p^2 \frac{\partial}{\partial t}(\eta v_y^{(2)}); \text{ and}$$
$$\psi_z = -\omega_p^2 \frac{\partial}{\partial t}(\eta v_z^{(2)}), \qquad (VI,1)$$

where

$$\eta = \frac{\Delta N(z,t)}{N_o} = \frac{\Delta N_m}{N_o} \cos\left\{\omega_p t - \omega_p n_{\underset{x}{o}}(\omega_p) z\right\} , \qquad (VI,2)$$

represents the charge bunching. If $\omega_T \neq 0$, η is linearly proportional to the pump field amplitude (II,5a). When $\omega_T = 0$, it is proportional to the square of this ([3], (IV,25)). Eq. (IV,4) and associated relations now yield the following coupled equations, viz.

$$\left\{\frac{\partial^2}{\partial t^2} p_{em}^2 \left(p_p^2\left[p_e^2 + \omega_p^2(1+\eta)\right] + \omega_T^2 p_e^2\right) + \omega_L^2 p_p^2 p_e^4\right\} v_y^{(2)} =$$

$$= -\omega_p^2 p_e^2 \frac{\partial}{\partial t}\left\{p_p^2 \omega_L \eta v_x^{(2)} - p_{em}^2 \omega_T \eta v_z^{(2)}\right\} . \qquad (VI,3)$$

$$\left\{\frac{\partial^2}{\partial t^2} + \omega_p^2(1+\eta)\right\} v_z^{(2)} = \omega_T \frac{\partial v_y^{(2)}}{\partial t} \qquad (VI,4)$$

$$\frac{\partial}{\partial t}\left\{p_e^2 + \omega_p^2(1+\eta)\right\} v_x^{(2)} = -\omega_L p_e^2 v_y^{(2)} . \qquad (VI,5)$$

The left hand side of (VI,4) is a Mathieu-equation, and the left hand side of (VI,5) can be reduced to one [3]. Both operators describe oscillations in a periodically perturbed, isotropic ionized medium

In order to study the instabilities in this medium, one would have to solve the coupled "Mathieu-type" equations, which is a very complicated matter. The spectral terms of the steady state can, of course, always be obtained from (VI,3), (VI,4), and (VI,5). One then obtains the (parametric) resonance conditions already described (IV,20). If the fundamental mode easily becomes non-linear, for example at the fourth reflection level, one can pump the medium with a higher frequency than 2ω, viz. 3ω, 4ω, etc., which correspond to the higher order Mathieu-instabilities. If losses are very small, as in the uppermost ionosphere, harmonic travelling wave pumping and generation of the more easily excited modes should be possible, provided the resonance conditions are satisfied. This interesting matter will be dealt with in a forthcoming report.

VII. Non-Linearities in the "Top-Side" Ionosphere

Even though the theory presented in the previous sections is based on the assumption that infinite, plane (magneto-ionic) waves travel in the system, which is not true within a wavelength or so of a "top-side" sounder, it should be possible to draw some general conclusions concerning the nature of the wave instabilities recorded by such a device.

At the normal total reflection, or self-oscillation levels, $X = 1-Y$, 1, and $1 + Y$, one only expects fundamental plasma "spikes", except at levels where any of the travelling wave resonances of (V,18) would take place. At these levels harmonic pumping should also be possible, i.e. plasma "spikes" could be recorded when the sounder emits at twice the resonance frequency.

Instabilities are likely to be very strong at the fourth reflection levels, at least as far as plane waves are concerned. These levels, which lie in the following Y^2-range (II,8), viz.

$$Y^2 = Y_R^{\,2} = \frac{1-X}{1-X\cos^2\theta}\,, \qquad (X<1)$$

$$Y^2 = Y_R^{\,2} = \frac{X-1}{X\cos^2\theta-1}\,, \qquad (X>1/\cos^2\theta)$$

depend upon the angle θ. Since the "top-side" sounder acts almost as a point source in the medium, plasma "spikes" at the $Y_R^{\,2}$-levels are only likely to be strong in wave normal directions for which

$$\frac{1}{n}\,\frac{dn}{d\theta} = 0\,,$$

which means that the Poynting vector is parallel to the wave normal. Since $\cos^2\theta$ is equal to 0, or 1, in these directions (longitudinal or transverse propagation), strong fundamental plasma "spikes" are only likely when

$$Y_R^{\,2} = 1,\ (\theta = 0),\ \text{and}\ Y_R^{\,2} = 1-X\ (\theta = \tfrac{\pi}{2}).$$

This agrees with the experimental results so far reported. Harmonic pumping (see VI) of these resonances should also be possible, especially for the one at $Y_R^{\,2} = 1$ ("cyclotron resonance"), since travelling wave resonances are easily obtained at this level (see (V,9)).

If $Y^2 = 1$, we obtain for small θ-values

$$n_x^{\,2}(\omega) \approx 1 + \frac{2(1-X)}{\sin^2\theta}\,, \qquad \text{(VII,1)}$$

which means that the transverse component of n_x is practically independent of θ (for small angles), or

$$n_x(\omega)\sin\theta \approx \sqrt{2(1-X)}\,. \qquad \text{(VII,2)}$$

Second harmonic travelling wave resonances (now shown in vector form) are thus easily obtainable in the direction around $\theta = 0$, as sketched in Fig. 3. The second harmonic pump wave continuously builds up forward and backward waves at the fundamental frequency which, in their turn, generates a backward second harmonic wave. The system becomes unstable at complete travelling wave resonance (see VI, and [3]), and "cyclotron spikes" will be recorded at both the fundamental and the second harmonic. Similar pumping schemes would also be effective at the higher harmonics, provided ν is small enough (as in the "top-side" ionosphere). Such harmonic "spikes", related to $Y_R^{\,2} = 1$, have been recorded by "top-side" sounders, but to the authors knowledge, none related to $Y^2 = 1-X$.

Unfortunately, the present "top-side" sounders are not equipped to record harmonics and fundamentals simultaneously. No doubt a recorder with such features would yield very interesting and important results and is almost a necessity if one wishes to study the top-side non-linearities thoroughly.

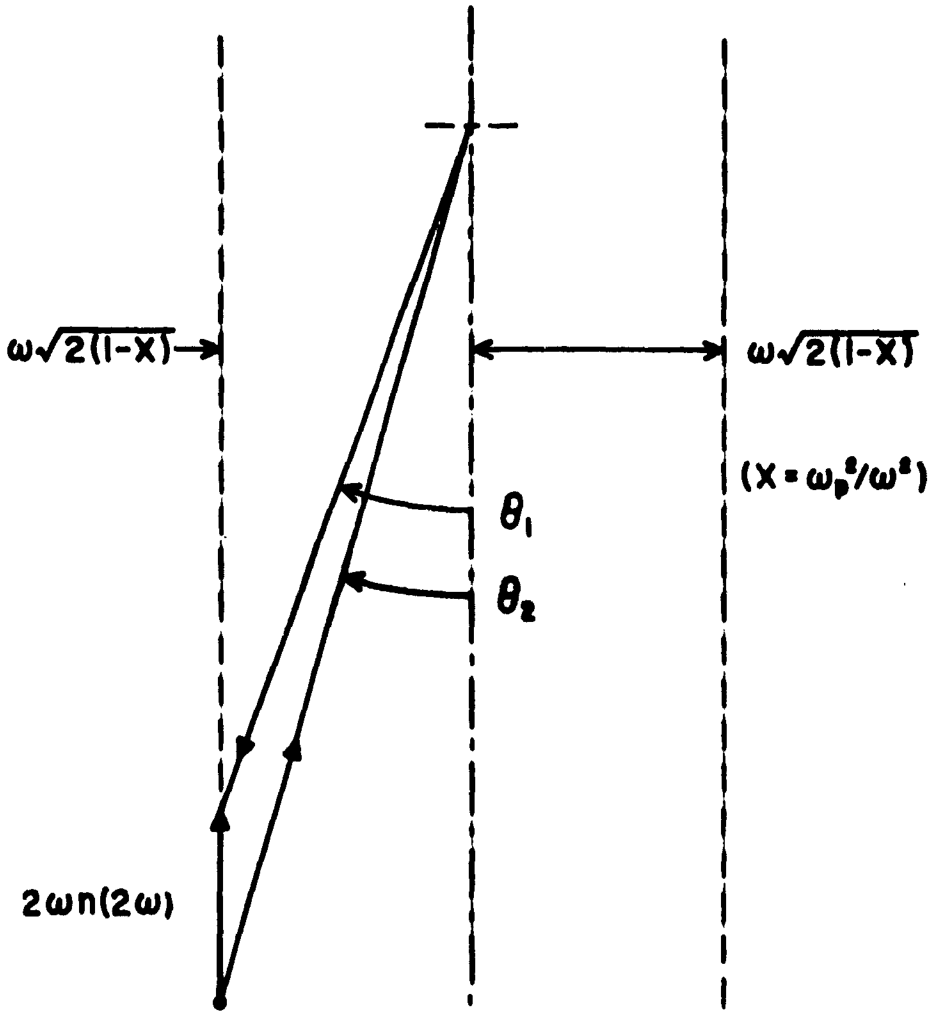


FIG. 3 DEMONSTRATING 2nd HARMONIC PUMPING AT $Y_R^2 = 1$

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